Fluid Dynamics Research 9 (1992) 155–164
North-Holland

FLUID DYNAMICS
RESEARCH

Displacement waves in saturated thermoelastic porous media. I. Basic equations

Jacob Bear

Albert and Anne Mansfield Chair in Water Resources, Technion – Israel Institute of Technology, Haifa 32000, Israel

Shaul Sorek

J. Blaustein Desert Research Institute, Ben-Gurion University, of the Negev, Sede Boker, Israel

Gabi Ben-Dor and Gedalia Mazor

Mechanical Engineering, Ben-Gurion University of the Negev, Beer Sheva, Israel

Received 24 January 1991; revised manuscript received 3 June 1991

Abstract. A complete set of macroscopic equations, the solution of which describes fluid and solid stresses, displacements and temperatures, evolving from an excitation of a saturated porous medium domain in the form of an abrupt pressure and temperature changes applied at the domain's boundary is presented. The fluid is a compressible Newtonian one and the solid is thermoelastic. Nonisothermal conditions prevail. The set of equations includes mass, momentum and energy balance equations, constitutive relations and definitions. A simple example indicates that the fluid and solid displacements are described by two coupled wave equations.

1. Introduction

Experimental studies conducted over the past two decades, e.g., Kormer et al. (1962), have indicated that when pressure shock waves propagate through a porous medium domain, or along the external boundary of such a domain, they are strongly attenuated. This phenomenon can be used to reduce damages caused by such waves to structures.

Inertial effects appear in the basic macroscopic momentum balance equation, for a fluid phase that occupies the entire void space. Usually, in modelling flow through porous media, it is assumed that the inertial forces are small relative to the viscous ones, and may, therefore, be neglected. Our objective in this paper is to construct a model describing single phase flow in a porous medium domain, in which the inertial effects play a major role.

Biot's (1956) work on compressional wave propagation was probably the first one that employed the fundamentals of porous medium mechanics. A large number of papers have appeared in the literature following Biot's pioneering work. A literature survey was presented by Corapcioglu (1989).

The complete model, presented at the macroscopic level, includes mass, momentum and energy balance equations for the fluid and for the solid skeleton, as well as constitutive relations that describe the behavior of the particular materials involved: a Newtonian compressible fluid that occupies the entire void space, and a deformable, thermoelastic solid matrix. Although the solid matrix is assumed to be deformable, its deformation, due to movement of solid particles, is assumed to be small.

In subsequent papers, the model developed here will be applied to particular cases of stress shock waves that travel in porous medium domains. Numerical solutions will be derived for these cases.

2. Mass balance equations

The macroscopic mass balance equations for the fluid and for the solid phases are obtained by averaging the corresponding microscopic equations over a representative elementary volume (REV). At the average, macroscopic level of description, each phase and component of a phase, and every extensive quantity of a phase or of a component, is regarded as a *continuum* that fills up the entire porous medium domain.

(a) *For the fluid.* Neglecting the dispersive flux of the fluid's mass, resulting from the fact that at the microscopic level both the densities and the velocities vary within the fluid's occupied portion of the representative elementary volume (REV), the macroscopic mass balance equation for the fluid, is

$$\frac{\partial \phi \rho_f}{\partial t} = -\nabla \cdot (\phi \rho_f V_f), \tag{1}$$

where ρ_α is the intrinsic phase average mass density of the α-phase ($\alpha = f, s$), V_α is the intrinsic phase average mass weighted velocity of the α-phase, and ϕ is the porosity.

(b) *For the solid.* Neglecting the dispersive flux of the solid's mass, the macroscopic mass balance equation for the solid is

$$\frac{\partial (1-\phi)\rho_s}{\partial t} = -\nabla \cdot [(1-\phi)\rho_s V_s], \tag{2}$$

or, since we have assumed negligible solid deformations, implying $\rho_s = \text{const.}$, in the equivalent form

$$\nabla \cdot V_s = -\frac{1}{1-\phi}\frac{D_s(1-\phi)}{Dt} = \frac{D_s \epsilon_{sk}}{Dt}, \tag{3}$$

where $D_\alpha/Dt \equiv \partial/\partial t + V_\alpha \cdot \nabla$ denotes the *material derivative*, which expresses the rate of change from the point of view of an observer travelling at the velocity V_α, and ϵ_{sk} denotes the (macroscopic) dilatation *of the solid skeleton*, i.e. of the porous medium as a whole. It is the first invariant of the skeleton's strain tensor, $\boldsymbol{\epsilon}_{sk}$.

The skeleton's strain can be expressed in terms of the (macroscopic, i.e., intrinsic phase average) displacement vector, $\boldsymbol{w}$, in the form

$$\epsilon_{skij} = \tfrac{1}{2}(\partial w_i/\partial x_j + \partial w_j/\partial x_i), \tag{4}$$

with the volumetric strain (= dilatation), ϵ_{sk}, expressed by

$$\epsilon_{sk} = \epsilon_{skii} = \nabla \cdot \boldsymbol{w}. \tag{5}$$

It may be of interest to note that if we start from the *microscopic equation of compatibility* for the solid phase,

$$\epsilon_{sij} = \tfrac{1}{2}(\partial w_{si}/\partial x_j + \partial w_{sj}/\partial x_i), \tag{6}$$

in which ϵ_s denotes the solid's (microscopic) strain, and $\boldsymbol{w}_s$ denotes the solid's (microscopic) displacement, and average it, we obtain

$$\overline{\epsilon_{sij}} = \tfrac{1}{2}\left(\overline{\frac{\partial w_{si}}{\partial x_j}} + \overline{\frac{\partial w_{sj}}{\partial x_i}}\right) = \tfrac{1}{2}\left(\frac{\partial \overline{w_{si}}}{\partial x_j} + \frac{\partial \overline{w_{sj}}}{\partial x_i}\right) + \frac{1}{U_0}\int_{S_{sf}} (w_{si}\nu_j + w_{sj}\nu_i)\, dS$$

$$\approx \tfrac{1}{2}\phi\left(\frac{\partial \overline{w_{si}}^{s}}{\partial x_j} + \frac{\partial \overline{w_{sj}}^{s}}{\partial x_i}\right), \tag{7}$$

where the overbar denotes the phase average and $\overline{(\ldots)_\alpha}^{\alpha}$ denotes the intrinsic phase average of $(\ldots)_\alpha$, with $\overline{(\ldots)_\alpha} = \phi\overline{(\ldots)_\alpha}^{\alpha}$. The approximation stems from the assumption

$$\frac{1}{U_0}\int_{S_{sf}} w_{si}\nu_j\, dS \approx \overline{w_{si}}^{s}\frac{1}{U_0}\int_{S_{sf}} \nu_j\, dS \equiv -\overline{w_{si}}^{s}\frac{\partial \phi}{\partial x_j}.$$

Hence, by comparison with (4), with $\boldsymbol{w} \equiv \overline{\boldsymbol{w}}^{s}$, we may conclude that

$$\overline{\epsilon_{sij}}^{s} = \frac{1}{2}\left(\frac{\partial \overline{w_{si}}^{s}}{\partial x_j} + \frac{\partial \overline{w_{sj}}^{s}}{\partial x_i}\right) \equiv \epsilon_{skij}. \tag{8}$$

(c) *For the porous medium as a whole.* By summing (1) and (2), we obtain the mass balance equation for the porous medium as a whole, in the form

$$\frac{\partial \rho_b}{\partial t} = -\nabla\cdot(\rho_b \boldsymbol{V}_s + \phi\rho_f \boldsymbol{V}_r), \tag{9}$$

where

$$\rho_b \equiv \phi\rho_f + (1-\phi)\rho_s \tag{10}$$

denotes the bulk density of the porous medium, and

$$\boldsymbol{V}_r \equiv \boldsymbol{V}_f - \boldsymbol{V}_s \tag{11}$$

denotes the velocity of the fluid relative to the solid. The use of the relative fluid velocity is of interest, because the averaged momentum balance of the fluid is usually expressed in terms of this velocity.

3. Momentum balance equations

Assuming that the mass and momentum dispersive fluxes are much smaller than their advective counterparts, and that the solid–fluid interface is a material surface with respect to the mass of both phases, the macroscopic momentum balance equations for a fluid phase, f, and for a solid one, s, are

$$\frac{\partial(\phi\rho_f \boldsymbol{V}_f)}{\partial t} = -\nabla\cdot\phi(\rho_f \boldsymbol{V}_f \boldsymbol{V}_f - \boldsymbol{\sigma}_f) + \phi\rho_f \boldsymbol{F} + \frac{1}{U_0}\int_{S_{fs}} \hat{\boldsymbol{\sigma}}_f\cdot\boldsymbol{\nu}_f\, dS, \tag{12}$$

$$\frac{\partial(1-\phi)\rho_s \boldsymbol{V}_s}{\partial t} = -\nabla\cdot(1-\phi)(\rho_s \boldsymbol{V}_s \boldsymbol{V}_s - \boldsymbol{\sigma}_s) + (1-\phi)\rho_s \boldsymbol{F} + \frac{1}{U_0}\int_{S_{sf}} \hat{\boldsymbol{\sigma}}_s\cdot\boldsymbol{\nu}_s\, dS, \tag{13}$$

respectively, where $\hat{\boldsymbol{\sigma}}_\alpha$ is the stress tensor of the α-phase, at the microscopic level, $\boldsymbol{\sigma}_\alpha$ is the intrinsic phase average stress tensor of the α-phase, $S_{\alpha\beta}$ is the microscopic surface separating the α-phase from all other (β-)phase(s), $\boldsymbol{\nu}_\alpha$ is the outward unit vector on $S_{\alpha\beta}$, $\boldsymbol{F}$ is the body force ($= -g\nabla z$), per unit mass, g is the gravity acceleration and z is the altitude.

We recall that these equations are obtained by averaging their microscopic counterparts. We note in them that the momentum, say of the fluid, per unit volume of porous medium, is given by $\phi\rho_f \boldsymbol{V}_f$. Also, the exchange of momentum between the two phases – a boundary condition at the microscopic level – is a momentum sink (per unit volume of porous medium) expressed by the surface integrals.

By summing the two equations, we eliminate the rate of exchange of momentum across the solid–fluid interface. We obtain the momentum balance equation for the porous medium as a whole, in the form

$$\frac{\partial}{\partial t}(\phi\rho_f \boldsymbol{V}_r + \rho_b \boldsymbol{V}_s) = -\nabla\cdot\left[\phi\rho_f(\boldsymbol{V}_r\boldsymbol{V}_r + \boldsymbol{V}_r\boldsymbol{V}_s + \boldsymbol{V}_s\boldsymbol{V}_r + \rho_b\boldsymbol{V}_s\boldsymbol{V}_s)\right] + \nabla\cdot\boldsymbol{\sigma}_t - \rho_b g\,\nabla z, \tag{14}$$

in which

$$\boldsymbol{\sigma}_t \equiv \phi\boldsymbol{\sigma}_f + (1-\phi)\boldsymbol{\sigma}_s \tag{15}$$

is the total (macroscopic) stress within the porous medium domain, and $\rho_b \boldsymbol{F}$ ($\equiv -\rho_b g\,\nabla z$) is the body force, per unit volume of porous medium, due to gravity.

For a fluid phase

$$\boldsymbol{\sigma}_f = \boldsymbol{\tau}_f - p\boldsymbol{I}, \tag{16}$$

where $\boldsymbol{\tau}_f$, and p denote the fluid's viscous stress and the pressure, respectively, and $\boldsymbol{I}$ is the unit tensor.

At this point we introduce Terzaghi's (1925) concept of *effective stress*, or *intergranular stress*, $\boldsymbol{\sigma}_s'$ (e.g., Bear and Bachmat, 1990). Essentially, this concept assumes that as the solid comprising the solid matrix (e.g., a granular material) is *almost* completely surrounded by an ambient fluid, the latter's stress, acting on the fluid–solid interface, produces a stress of equal magnitude in the solid (say, within each individual grain), without contributing to the solid matrix's deformation. The latter is produced mainly by forces that (in a granular material) are transmitted from grain to grain at contact points. Accordingly, the *strain producing stress*, or the *intergranular stress* (effective stress) is obtained by subtracting the stress in the fluid (or pressure, if shear stress is neglected) from the stress in the solid, with both stresses are at the average level, and taking stress positive for tension and pressure in a fluid positive for compression. With this definition of effective stress, we may express the total stress in the form

$$\boldsymbol{\sigma}_t = \boldsymbol{\sigma}_s' + \boldsymbol{\sigma}_f, \tag{17}$$

with the effective stress defined by

$$\boldsymbol{\sigma}_s' = (1-\phi)(\boldsymbol{\sigma}_s - \boldsymbol{\sigma}_f), \quad \text{with } \sigma_{sij}' = \sigma_{sji}'. \tag{18}$$

Note that $\boldsymbol{\sigma}_s$ and $\boldsymbol{\sigma}_f$ are intrinsic averages. In a one-dimensional case, they may be defined as forces per unit area of the respective phase. On the other hand, $\boldsymbol{\sigma}_t$ and $\boldsymbol{\sigma}_s'$ should be interpreted as per unit area of the porous medium as a whole.

The macroscopic momentum balance equation for the fluid alone, (12), written in indicial notation, can be developed to the form (Bear and Bachmat, 1990)

$$\begin{aligned}
&\frac{\partial}{\partial t}\phi\rho_{\mathrm{f}}(V_{\mathrm{r}i}+V_{\mathrm{s}i}) \\
&\quad = -\frac{\partial}{\partial x_j}\left[\phi\rho_{\mathrm{f}}(V_{\mathrm{r}i}V_{\mathrm{r}j}+V_{\mathrm{r}i}V_{\mathrm{s}j}+V_{\mathrm{r}j}V_{\mathrm{s}i}+V_{\mathrm{s}i}V_{\mathrm{s}j})\right] \\
&\qquad -\phi\left(\frac{\partial p}{\partial x_j}+\rho_{\mathrm{f}}g\frac{\partial z}{\partial x_j}\right)T_{ji}^{*}+\mu_{\mathrm{f}}\left[\frac{\partial^2\phi V_{\mathrm{r}i}}{\partial x_j\,\partial x_j}+\frac{\partial}{\partial x_j}\left(\phi\frac{\partial V_{\mathrm{s}i}}{\partial x_j}\right)\right] \\
&\qquad +(\mu_{\mathrm{f}}+\lambda_{\mathrm{f}}'')\left[\frac{\partial^2\phi V_{\mathrm{r}j}}{\partial x_i\,\partial x_j}-\frac{1}{\phi}\left(\frac{\partial\phi}{\partial t}+\frac{\partial\phi(V_{\mathrm{r}j}+V_{\mathrm{s}j})}{\partial x_j}\right)\frac{\partial\phi}{\partial x_i}+\frac{\partial}{\partial x_i}\left(\phi\frac{\partial V_{\mathrm{s}j}}{\partial x_j}\right)\right] \\
&\qquad -\mu_{\mathrm{f}}\alpha_{ij}\frac{C_{\mathrm{v}}}{\Delta_{\mathrm{f}}^2}\phi V_{\mathrm{r}j},
\end{aligned} \tag{19}$$

where T_{ij}^{*} and α_{ij} are tensorial coefficients that constitute macroscopic representations of the microscopic configuration of the fluid–solid interface, S_{fs}, within the REV, μ_{f} is the fluid's viscosity, $\lambda_{\mathrm{f}}''+\frac{2}{3}\mu_{\mathrm{f}}$ is the fluid's second viscosity, C_{v} is the macroscopic dimensionless shape factor, Δ_{f} is the hydraulic radius of the fluid occupied void space. Both μ_{f} and λ_{f}'' are intrinsic phase averages of the respective microscopic coefficients.

The coefficient $\mathbf{T}_{\alpha}^{*}$, $\alpha=\mathrm{f},\mathrm{s}$, a second rank symmetric tensor, may be interpreted as the *tortuosity* of the α-phase within the REV. Here, it transforms the local body force into a macroscopic one. The coefficient α_{ij} introduces the effect of the configuration of the solid–fluid surface in the term that transforms part of the force resisting the flow at a point to an averaged resistance force at the fluid–solid interface.

The constitutive equation employed in deriving the averaged momentum balance equation (19), for a compressible Newtonian fluid, will be presented at a later stage. The last term on the rhs of (19) expresses the drag, due to momentum transfer at the fluid–solid interface, i.e.

$$\frac{1}{U_0}\int_{S_{\mathrm{fs}}}\tau_{ij}\nu_j\,\mathrm{d}S=-\mu\frac{C_{\mathrm{v}}}{\Delta_{\mathrm{f}}^2}\alpha_{ij}q_{\mathrm{r}j}.$$

4. Energy balance equations

In writing the macroscopic energy balance equations for the fluid and for the solid, we assume reversible processes, and constant specific heat at constant volume, C_{f} for the fluid and constant specific heat at constant strain, C_{s} for the solid. We shall omit external energy sources, the effect of changes in component concentrations and the energy associated with viscous dissipation, i.e. $|\boldsymbol{\tau}:\nabla V_{\mathrm{f}}| \ll |p\,\boldsymbol{I}\nabla\cdot V_{\mathrm{s}}|$. If necessary, these can be added at a later stage.

Subject to certain simplifying assumptions, Bear and Bachmat (1990) present the following macroscopic energy balance equations. For the fluid phase,

$$\begin{aligned}
\frac{\partial}{\partial t}(\phi\rho_{\mathrm{f}}C_{\mathrm{f}}T_{\mathrm{f}}) &= -\nabla\cdot\phi\left[\rho_{\mathrm{f}}C_{\mathrm{f}}T_{\mathrm{f}}(V_{\mathrm{r}}+V_{\mathrm{s}})+\boldsymbol{J}_{\mathrm{f}}^{\mathrm{H}}+\boldsymbol{J}_{\mathrm{f}}^{*\mathrm{H}}\right] \\
&\quad -f_{\mathrm{f}\to\mathrm{s}}^{\mathrm{H}}-T_{\mathrm{f}}\beta_T/\beta_p\,|_{\rho_{\mathrm{f}}}\nabla\cdot\phi(V_{\mathrm{r}}+V_{\mathrm{s}}),
\end{aligned} \tag{20}$$

where J_α^{H} is the conductive heat flux in the α-phase, at the macroscopic level, $J_\alpha^{*\mathrm{H}}$ is the dispersive heat flux in the α-phase, $f_{\alpha\to\beta}^{\mathrm{H}}$ is the rate of heat transferred from the α-phase to the β-phase, across their common interface within the REV, per unit volume of the β-phase, the β_p is the coefficient of fluid compressibility at constant temperature and the β_T is the coefficient of fluid thermal (volumetric) expansion at constant pressure, with

$$\beta_p = \frac{1}{\rho_{\mathrm{f}}}\frac{\partial \rho_{\mathrm{f}}}{\partial p_{\mathrm{f}}}\bigg|_{T_{\mathrm{f}}}, \quad \beta_T = -\frac{1}{\rho_{\mathrm{f}}}\frac{\partial \rho_{\mathrm{f}}}{\partial T_{\mathrm{f}}}\bigg|_{p_{\mathrm{f}}}. \tag{21}$$

By writing the macroscopic energy balance equation in this particular form, and recalling that $V_{\mathrm{r}} + V_{\mathrm{s}} \equiv V_{\mathrm{f}}$, its physical interpretation is clear. The product $\phi\rho_{\mathrm{f}}C_{\mathrm{f}}T_{\mathrm{f}}$ expresses the internal energy per unit volume of porous medium. Thus, the lhs epresses the rate of accumulation of internal energy per unit volume of porous medium. This accumulation is due to (a) excess of inflow over outflow of energy per unit volume and unit time, expressed by minus divergence of the total (macroscopic) energy flux, which, in turn, is composed of an advective flux, a dispersive flux and a diffusive (≡ conductive) one, (b) an energy sink, per unit volume of porous medium and per unit time, due to the transfer of energy from the considered phase to all surrounding phases, across their common microscopic interfaces, and (c) energy gain, per unit volume of porous medium and unit time, due to the fluid's compression.

Since we consider here the case $T_{\mathrm{f}} \neq T_{\mathrm{s}}$, we may express the rate of heat transfer, $f_{\mathrm{s}\to\mathrm{f}}^{\mathrm{H}}$, by

$$f_{\mathrm{s}\to\mathrm{f}}^{\mathrm{H}} = \alpha^{*H}(T_{\mathrm{s}} - T_{\mathrm{f}}), \tag{22}$$

where $\alpha^{*\mathrm{H}}$ is a heat transfer coefficient.

For a *thermo-elastic* isotropic solid phase, the macroscopic heat balance equation is given by

$$(\partial/\partial t)(1-\phi)\rho_{\mathrm{s}}C_{\mathrm{s}}T_{\mathrm{s}} = -\nabla\cdot(1-\phi)\left(\rho_{\mathrm{s}}C_{\mathrm{s}}T_{\mathrm{s}}V_{\mathrm{s}} + J_{\mathrm{s}}^{\mathrm{H}}\right) - f_{\mathrm{s}\to\mathrm{f}}^{\mathrm{H}} - T_{\mathrm{s}}\bar{\eta}\,\mathrm{D}_{\mathrm{s}}\epsilon_{\mathrm{sk}}/\mathrm{D}t, \tag{23}$$

where $\bar{\eta}$ $(= (1-\phi)\bar{\eta}^{\mathrm{s}})$ is a macroscopic coefficient that appears in the constitutive equation for the solid matrix. In writing the last term on the rhs of (23), we have assumed that $\nabla\cdot V_{\mathrm{s}} = 0$, or $\rho_{\mathrm{s}} = \text{const.}$, at the microscopic level (i.e. isochoric motion).

For an isotropic thermoelastic solid

$$\eta\delta_{ij} = -\partial\sigma_{\mathrm{s}ij}/\partial T_{\mathrm{s}}\,|_{\epsilon_{\mathrm{sk}}=\text{const.}}$$

We note that in (20) and (23), we have

$$f_{\mathrm{s}\to\mathrm{f}}^{\mathrm{H}} = -f_{\mathrm{f}\to\mathrm{s}}^{\mathrm{H}}.$$

In order to express these balance equations in terms of average temperatures as the only state variables, we have to introduce appropriate expressions for the macroscopic conductive fluxes, $J_{\mathrm{f}}^{\mathrm{H}}$ and $J_{\mathrm{s}}^{\mathrm{H}}$, as well as for the terms that express the (average) rate of exchange of heat between the phases. These topics are discussed below.

The macroscopic conductive heat fluxes, $J_{\mathrm{f}}^{\mathrm{H}}$ and $J_{\mathrm{s}}^{\mathrm{H}}$, for the fluid and for the solid phases, respectively, for the general case of $T_{\mathrm{f}} \neq T_{\mathrm{s}}$, are obtained by averaging the corresponding microscopic flux equations, taking into account the possible exchange of energy between the phases. Thus (Bear and Bachmat, 1990)

$$\begin{aligned} J_{\alpha i}^{\mathrm{H}} &\equiv -\lambda_\alpha \overline{\frac{\partial T_\alpha}{\partial x_i}}^{\alpha} \\ &= -\frac{\lambda_\alpha}{\lambda_\alpha - \lambda_\beta}\left[\left(\lambda_\alpha\frac{\partial \overline{T_\alpha}^{\alpha}}{\partial x_i} - \lambda_\beta\frac{\partial \overline{T_\beta}^{\beta}}{\partial x_i}\right)T_{\alpha ij}^{*} - \frac{\lambda_\beta}{\theta_\alpha}\frac{\partial}{\partial x_j}\theta_\alpha\left(\overline{T_\alpha}^{\alpha} - \overline{T_\beta}^{\beta}\right)\right], \end{aligned} \tag{24}$$

where λ_α denotes the thermal conductivity of the α-phase, $\alpha, \beta = \mathrm{f}, \mathrm{s}$, with $\theta_\alpha + \theta_\beta = 1$, and $\phi T^*_{\alpha ij} + (1-\phi) T^*_{\beta ij} = \delta_{ij}$. We note the effect of heat transfer between the two phases, whenever they are not at the same averaged temperature.

When these temperatures are the same, the above expression reduces to

$$J^{\mathrm{H}}_{\alpha i} = -\lambda_\alpha \overline{\frac{\partial T_\alpha}{\partial x_i}}^{\alpha} = -\lambda_\alpha T^*_{\alpha ij} \frac{\partial \overline{T_\alpha}^{\alpha}}{\partial x_j}. \tag{25}$$

We may interpret (24) as indicating that in the case of $T_\mathrm{f} \neq T_\mathrm{s}$, the tortuosity of an α-phase is not a constant, but depends on both the contrast between the thermal conductivities of the two phases, and the difference in temperatures between them.

The fluid's dispersive heat flux can be expressed in the form (Bear and Bachmat, 1990)

$$\boldsymbol{J}^{*\mathrm{H}}_{\mathrm{f}} = -\mathbf{D}^{\mathrm{H}} \cdot \nabla \cdot (\rho C_\mathrm{f} T_\mathrm{f}), \tag{26}$$

where $\mathbf{D}^{\mathrm{H}}_{\mathrm{f}}$, a second rank symmetric tensor, denotes the fluid's coefficient of thermal dispersion. In most practical situations, thermal dispersion can be neglected with respect to thermal advection.

5. Constitutive relations

These express stress–strain relations for particular (fluid and solid) materials, and diffusive fluxes of particular extensive quantities. Obviously, we are interested here in these relations at the macroscopic level. They can be obtained by averaging their microscopic counterparts over an REV.

However, subject to certain simplifying assumptions, the averaged relations take on forms that are similar to their microscopic counterparts, except that the coefficients that appear in them have values (to be determined experimentally) that are different from the corresponding microscopic ones.

Stress–strain relationship for an isotropic thermoelastic solid matrix that undergoes small deformations. At the microscopic level, for a thermoelastic solid, this relationship, obtained from experiments, takes the form

$$\sigma_{\mathrm{s}ij} = \lambda''_\mathrm{s} \frac{\partial w_{\mathrm{s}l}}{\partial x_l} \delta_{ij} + \mu'_\mathrm{s} \left(\frac{\partial w_{\mathrm{s}i}}{\partial x_j} + \frac{\partial w_{\mathrm{s}j}}{\partial x_i} \right) - \eta (T_\mathrm{s} - T_{\mathrm{s}0}) \delta_{ij}, \tag{27}$$

where λ''_s and μ'_s are the Lamé constants, and η is the coefficient of thermal stress of the solid. For the Lamé constants, we may write

$$\lambda''_\mathrm{s} = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu'_\mathrm{s} = \frac{E}{2(1+\nu)},$$

where ν is Poisson's coefficient, E is the modulus of elasticity, δ is Kronecker's delta, η is a thermal stress coefficient of the solid ($= \alpha E/(1-2\nu)$), α is a coefficient of linear thermal expansion of the solid. At the macroscopic level, the constitutive relationship for a solid matrix, has the general form

$$\sigma'_{\mathrm{s}ij} = \sigma'_{\mathrm{s}ij}(\epsilon, \dot{\epsilon}, T).$$

The detailed macroscopic constitutive relation for a solid matrix is obtained by averaging the microscopic one (27). By performing this average and neglecting certain terms, primarily those

associated with surface integrals, and assuming that the solid matrix is isotropic, we obtain a macroscopic relationship that is similar in shape to its microscopic counterpart, (27), viz.

$$\sigma'_{sij} = \overline{\lambda}''_s \frac{\partial w_{sl}}{\partial x_l}\delta_{ij} + \overline{\mu}'_s\left(\frac{\partial w_{si}}{\partial x_j} + \frac{\partial w_{sj}}{\partial x_i}\right) - \overline{\eta}(T_s - T_{s0})\delta_{ij}, \tag{28}$$

where $\boldsymbol{w}$ and T_s are intrinsic phase averages, and $\overline{\lambda}''_s$, $\overline{\mu}'_s$ and $\overline{\eta}$ are now constants that have to be determined experimentally. They are usually referred to as *Lamé coefficients of the solid matrix*. They are analogous to the Lamé constants that appear in the microscopic stress-strain relation.

In view of (8), we can rewrite (28) in terms of the macroscopic strain, ϵ_{sk}, in the form

$$\sigma'_{sij} = 2\overline{\mu}'_s\epsilon_{skij} + \overline{\lambda}''_s\epsilon_{sk}\delta_{ij} - \overline{\eta}(T_s - T_{s0})\delta_{ij}, \tag{29}$$

in which we note the effective stress on the lhs.

Stress-rate of strain relationship for a Newtonian compressible fluid. At the microscopic level, this constitutive relationship takes the form

$$\tau_{fij} = \mu_f\left(\frac{\partial V_{fi}}{\partial x_j} + \frac{\partial V_{fj}}{\partial x_i}\right) + \lambda''_f\frac{\partial V_{fk}}{\partial x_k}\delta_{ij}. \tag{30}$$

By averaging this equation, assuming

$$V_i \mid_{S_{fs}} = V_{si} \mid_{S_{fs}} = \tilde{V}^{fs}_{si} = \overline{V_{si}}^{s}, \quad \hat{V}^{fs}_{si} \equiv \frac{1}{S_{fs}}\int_{S_{sf}} V_{si}\, dS,$$

we obtain

$$\overline{\tau_{fij}} = \overline{\mu}^{f}_{f}\left(\frac{\partial\phi V_{ri}}{\partial x_j} + \frac{\partial\phi V_{rj}}{\partial x_i}\right) + \overline{\lambda}''^{f}_{f}\frac{\partial\phi V_{rk}}{\partial x_k}\delta_{ij} + 2\phi\overline{\mu}^{f}_{f}\frac{D_s\epsilon_{skij}}{Dt} + \phi\overline{\lambda}''^{f}_{f}\frac{D_s\epsilon_{skij}}{Dt}. \tag{31}$$

Fluid compressibility.

$$\rho_f = \rho_f(p, T_f). \tag{32}$$

Table 1
Summary of balance equations and constitutive relations for saturated flow of a compressible Newtonian fluid in an isotropic linearly thermoelastic porous medium

Equation	Description	Dependent variables
(2)	solid's mass balance equation	V_s, ϕ
(9)	mass balance for porous medium	ρ_f, ρ_b, V_r
(14)	momentum balance for porous medium	$\boldsymbol{\sigma}_t$
(19)	fluid's momentum balance	p
(20)	fluid's energy balance	T_f
(23)	solid's energy balance	T_s, ϵ_{sk}
(28)	solid matrix constitutive relation	$\boldsymbol{\sigma}'_s$, $\boldsymbol{w}$
(31)	fluid's constitutive relation	τ
(32)	fluid's equation of state	
(4)	skeleton's compatibility equation	
(5)	volumetric strain (dilatation)	
(17)	definition of effective stress	$\boldsymbol{\sigma}_f$
(10)	density of porous medium	
(16)	fluid's stress	
	total: 14 equations	14 variables

6. Summary of equations and variables

At this stage, we have stated all the equations, balance equations, constitutive relations and definitions, included in our model. Altogether, the state variables and the equation are listed in table 1.

In a sequel paper, following Bear and Sorek (1990), we intend to rewrite the above model for various sets of restricting conditions, eventually solving it for a number of cases of practical interest. We intend to focus on the pressure, p, the displacements, $\boldsymbol{w}_f$, $\boldsymbol{w}_s$ and the temperatures, T_f and T_s, as the major variables.

7. A simple case

To obtain some feeling for the nature of the changes in $\boldsymbol{w}_f(\boldsymbol{x}, t)$ and $\boldsymbol{w}_s(\boldsymbol{x}, t)$, in addition to the assumptions underlying the above development, let us simplify the above (macroscopic) model by assuming:

(i) isothermal conditions.

(ii) Negligible vorticity in both phases (at the macroscopic level), i.e. $|V_{\alpha j}(\partial V_{\alpha i}/\partial x_j - \partial V_{\alpha j}/\partial x_i)| \ll |(\partial/\partial x_i)V_\alpha^2/2|$.

(iii) $\boldsymbol{V}_\alpha \equiv \mathrm{D}_s\boldsymbol{w}_\alpha/\mathrm{D}t \approx \partial \boldsymbol{w}_\alpha/\partial t$.

(iv) The viscous drag within the fluid is much smaller than that at the solid–fluid interface. Mathematically, this assumption is expressed as $|\partial\overline{\phi\tau_{ij}}^{f}/\partial x_j| \ll |\mathrm{U}_0^{-1}\int_{S_{sf}}\tau_{ij}\nu_{fj}\,\mathrm{d}S|$.

Under these assumptions, the momentum balance for the fluid, when accounting for the mass balance equation, reduces to form

$$\rho_f\frac{\partial^2\boldsymbol{w}_f}{\partial t^2} + \rho_f\nabla V_f^2/2 = -(\nabla p + \rho_f g\ \nabla z)\mathbf{T}_f^* - \mu_f\boldsymbol{\alpha}\frac{C_v}{\Delta_f^2}\frac{\partial}{\partial t}(\boldsymbol{w}_f - \boldsymbol{w}_s). \tag{33}$$

For the solid phase, by subtracting this equation from that written for the momentum balance equation written for the porous medium as a whole, taking into account the mass balance equations, we obtain

$$(1-\phi)\rho_s\frac{\partial^2\boldsymbol{w}_s}{\partial t^2} + (1-\phi)\rho_s\nabla\left(V_s^2/2\right) = \nabla\cdot\boldsymbol{\sigma}_t - \rho_b g\ \nabla z + \phi(\nabla p + \rho_f g\ \nabla z)\mathbf{T}_f^* + \mu_f\boldsymbol{\alpha}\frac{C_v}{\Delta_f^2}\frac{\partial}{\partial t}(\boldsymbol{w}_f - \boldsymbol{w}_s). \tag{34}$$

We note that (33) and (34) are wave equations. They can be rewritten as nonlinear wave equations in terms of the phase velocities. Let us demonstrate this fact in the special case of vertical one-dimensional domain.

Let us first introduce the definition of piezometric head (energy per unit weight) for a compressible fluid

$$h^* = z + \int_0^p \frac{\mathrm{d}p}{g\rho_f(p)}, \tag{35}$$

also known as *Hubbert's potential*, with

$$\frac{\partial h^*}{\partial t} = \frac{1}{g\rho_f}\frac{\partial p}{\partial t}, \quad \nabla h^* = \frac{1}{g\rho_f}\nabla p + \nabla z. \tag{36}$$

For the total stress in the porous medium, $\boldsymbol{\sigma}_t$, we have in a one-dimensional vertical domain $\mathrm{d}\sigma_t = g\rho_b\,\mathrm{d}z$.

With these definitions, we may rewrite (33) in the form

$$\frac{\partial V_f}{\partial t} + V_f \frac{\partial V_f}{\partial z} = \mathscr{F}_f, \tag{37}$$

in which

$$\mathscr{F}_f \equiv -\left[g \frac{\partial h^*}{\partial z} T^* + \frac{\mu_f \alpha}{\rho_f} \frac{C_v}{\Delta_f^2} (V_f - V_s) \right].$$

By differentiating (37) with respect to time, and replacing in it the term $\partial V_f/\partial t$ from (37), we obtain

$$\frac{\partial^2 V_f}{\partial t^2} - V_f^2 \frac{\partial^2 V_f}{\partial z^2} = -\frac{\partial V_f}{\partial z}\left(\mathscr{F}_f - V_f \frac{\partial V_f}{\partial z} \right) = -V_f \frac{\partial \mathscr{F}_f}{\partial z} + V_f \left(\frac{\partial V_f}{\partial z} \right)^2 + \frac{\partial \mathscr{F}_f}{\partial t}. \tag{38}$$

With

$$\mathscr{F}_s \equiv \frac{\phi g \rho_f}{(1-\phi)\rho_s} \frac{\partial h^*}{\partial z} T^* + \frac{\mu_f \alpha}{(1-\phi)\rho_s} \frac{C_v}{\Delta_f^2} (V_f - V_s),$$

the equation for the solid is analogous to (38), in which subscript f is replaced by subscript s.

In this way, we have demonstrated, albeit for a one-dimensional case, that (33) and (34) can be transformed into wave equations for the phase velocities.

Obviously, a complete model will still include appropriate constitutive equations and equations of state.

8. Conclusions

We have presented a complete set of macroscopic equations that describe fluid pressure changes and fluid and solid temperatures and displacements, in a saturated thermoelastic porous medium domain. The set includes mass, momentum and energy balance equations, constitutive relations and definitions. Included in the balance equations are inertial effects, as well as drag within the fluid and energy sources due to fluid compressibility and solid matrix deformability.

A simple example was presented in order to show that the solid and fluid displacements obey coupled wave equations.

References

Bear, J. and Y. Bachmat (1990) *Introduction to Modeling of Transport Phenomena in Porous Media* (Kluwer, Dordrecht).

Bear, J. and S. Sorek (1990) Evolution of mass and momentum balances following an abrupt pressure impact in a porous medium, *Transport Porous Media 5*, 169–196.

Biot, M.A. (1956) Theory of propagation of elastic waves in fluid-saturated porous solid, *J. Accous. Soc. Am. 28* 168–191.

Corapcioglu, M.Y. (1989) Wave Propagation in porous media–a review, in: *Transport Processes in Porous Media*, eds. J. Bear and M.Y. Corapcioglu (Reidel, Dordrecht).

Kormer, S.B., Funtikov, A.I., Urlin, V.D. and A.N Kolesinkova (1962) *Soviet Phys. JETP 15*, 477–488.

Terzaghi, K. (1925) *Erdbaumechanik auf Bodenphysikalische Grundlage* (Deuticke, Leipzig).